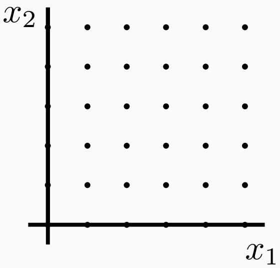
LP-based approximation methods

March 19, 2019

Linear inequalities

Example:

$$x_1 + 2 \cdot x_2 \leq 8$$
$$2 \cdot x_1 + x_2 \leq 6$$
$$x_1, x_2 \geq 0$$



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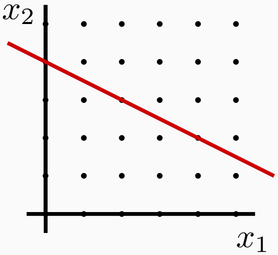

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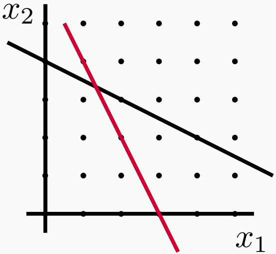

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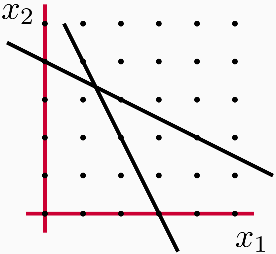

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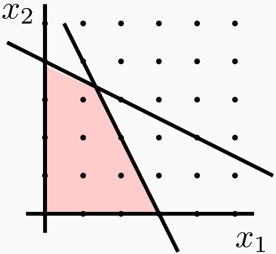

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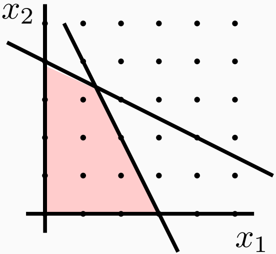

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Maximize/minimize a linear function over the set of solutions.

Example: $\max\{x_1 + x_2\}$.

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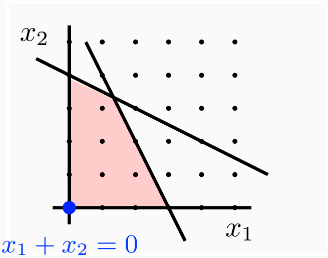

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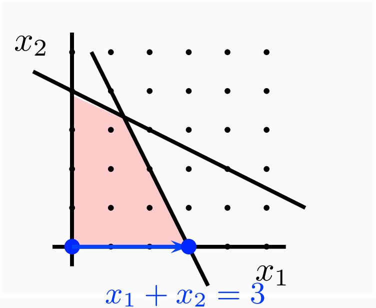

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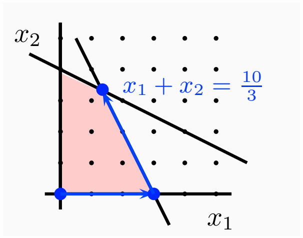

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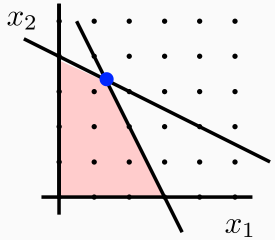

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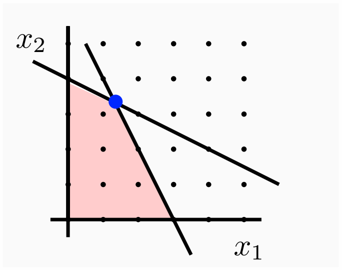

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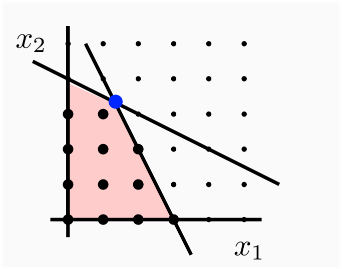

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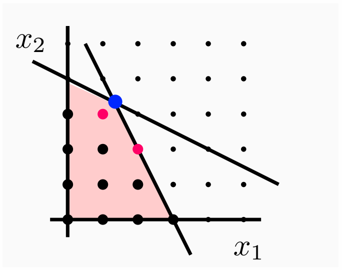

Another example

Example:

$$x_1 + 10 \cdot x_2 \le 10$$

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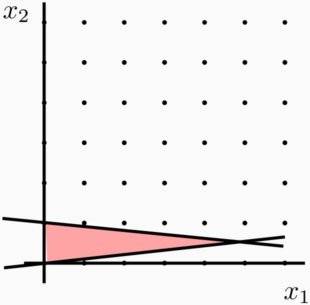
$$\max\{x_1\}$$



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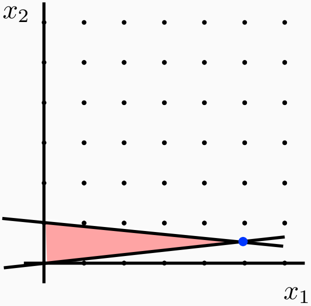

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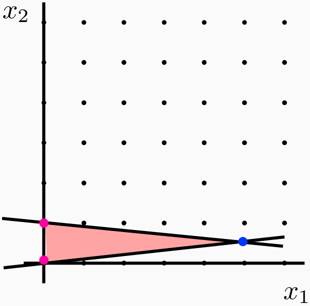

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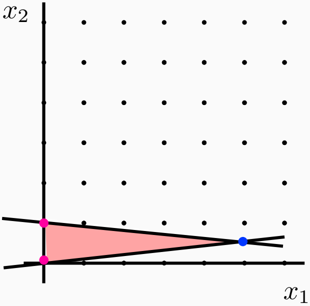

The fractional optimum can be far from the integer one.

Approaches

Bad news: integer programming is NP-complete.

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Good news: there exist efficient algorithms.

- totally unimodular matrices
  - every square submatrix has determinant 0, +1 or -1
- cutting plane methods
  - adding further inequalities that separate the actual optimum from the convex hull of the true feasible set
- branch and bound methods
  - systematically enumerating the candidate solutions, forming a rooted tree
- heuristic methods (tabu search, hill climbing, simulated annealing, ant colony optimization, etc)
  - some would call these 'voodoo'...

Approximation factor

Given a minimization problem, an α-approximation algorithm provides a solution of value at most $\alpha \cdot OPT$ (for maximization problems, of value at least OPT/α).

That is, the optimal solution is always guaranteed to be within a (predetermined) multiplicative factor of the returned solution.

Vertex cover

Problem

Given a graph $G = (V, E)$, find a minimum number of vertices covering every edge.

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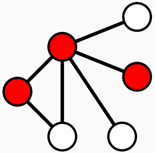

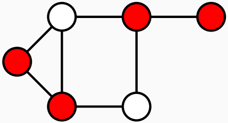

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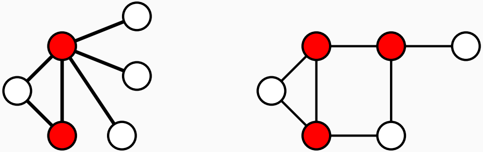

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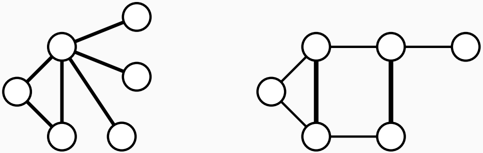

Simple algorithm:

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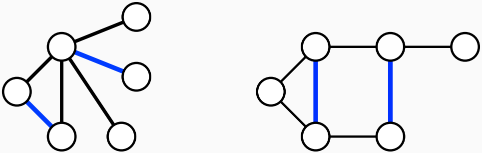

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Step 1. Take an inclusionwise maximal matching M.

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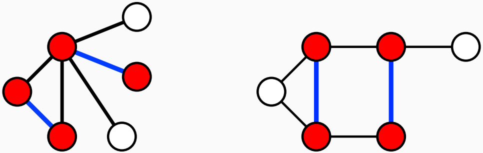

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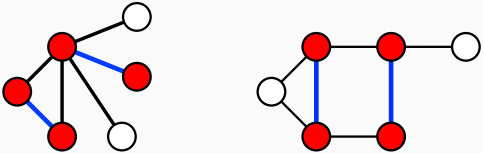

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Observation

This gives a 2-approximation.

Vertex cover - complexity

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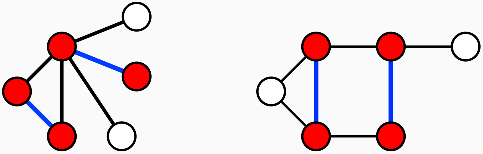

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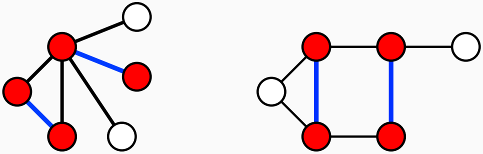

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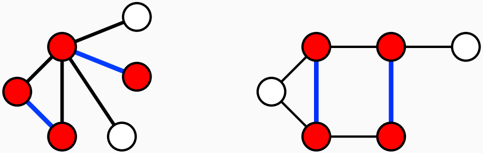

- One of Karp's 21 NP-complete problems.
- Moreover, it is APX-complete.

  - No better than 1.3606-approximation unless $\mathbf{P} = \mathbf{NP}$.

  - No better than 2-approximation assuming the Unique Games Conjecture.

Idea

Integer program

$$\min c^T \cdot x$$
$$A \cdot x \leq b$$
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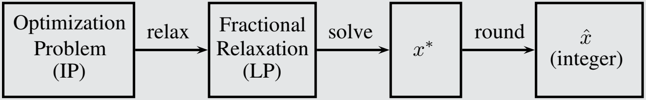
- the solution may not be optimal — Approximation?

Analysing the solution



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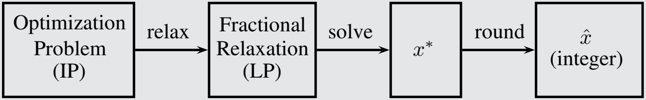

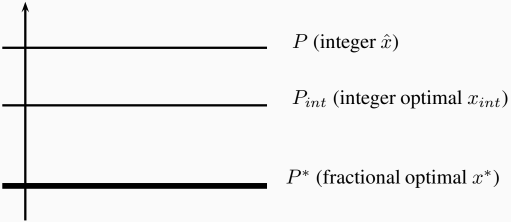

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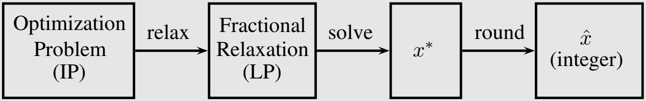

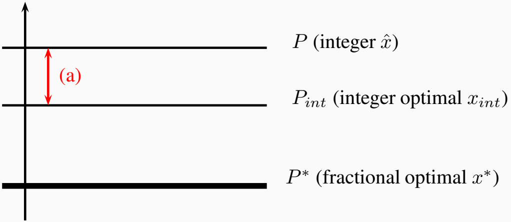

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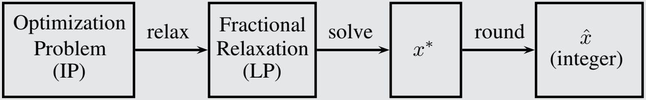

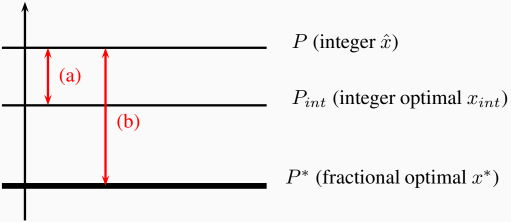

(a) = Approximation ratio between $\hat{x}$ and x_{int}.

(b) = Approximation ration between $\hat{x}$ and x^*.

Vertex cover revisited

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IP formulation

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\begin{aligned}
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LP relaxation

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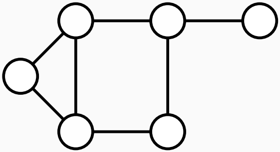

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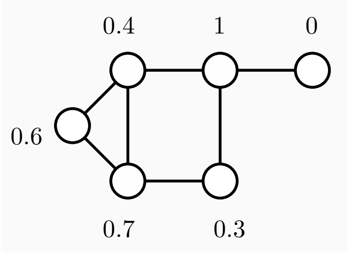

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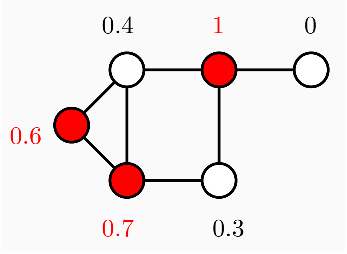

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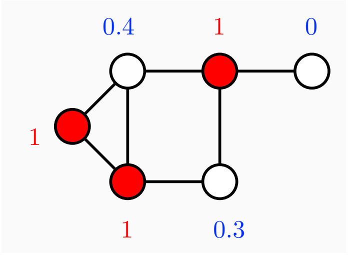

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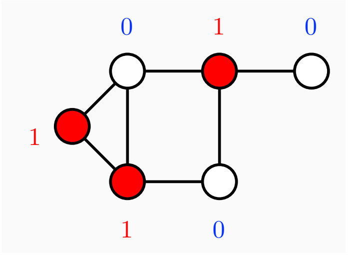

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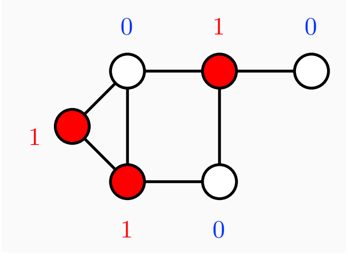

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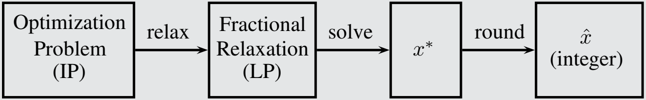
Proof

- $\hat{x}$ is integer
- $\hat{x}$ is feasible
- $\hat{x}_v \leq 2 \cdot x_v^*$, hence

$$\sum_{v \in V} \hat{x}_v \leq 2 \cdot \sum_{v \in V} x_v^*$$
$$\leq 2 \cdot OPT$$

Iteration

Treshold rounding



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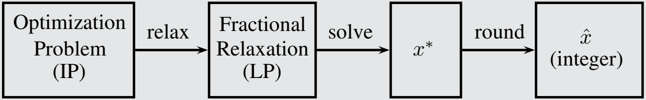
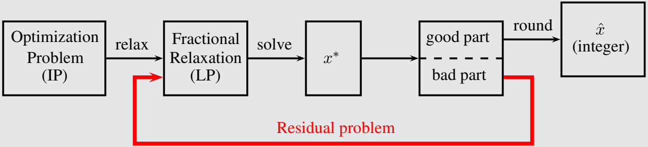


Iterative rounding



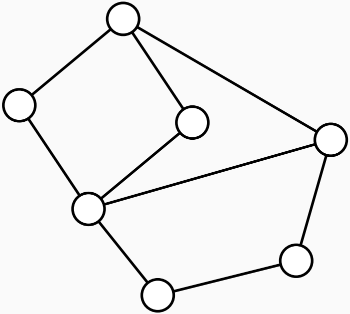
Maximum bipartite matching

LP formulation

$$\max \sum_{e \in E} x_e$$

$$\sum_{w \in V} x_{vw} \leq 1 \quad \text{for } v \in V$$

$$x_e \geq 0 \quad \text{for } e \in E$$



Maximum bipartite matching

LP formulation

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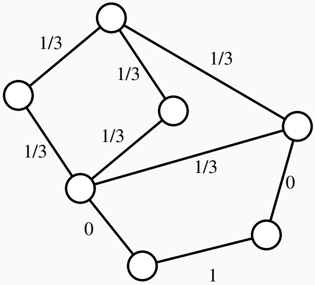

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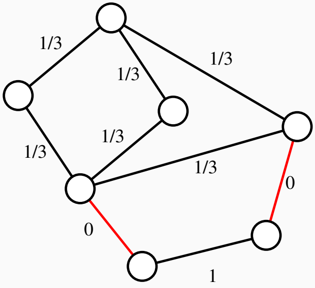

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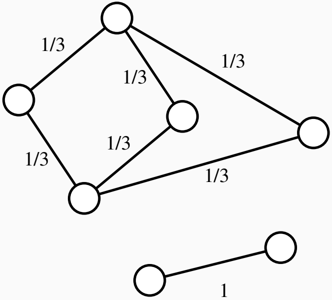

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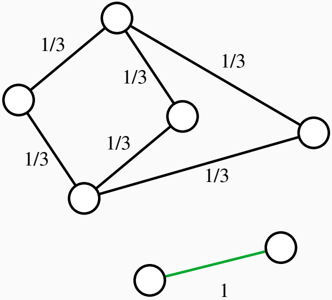

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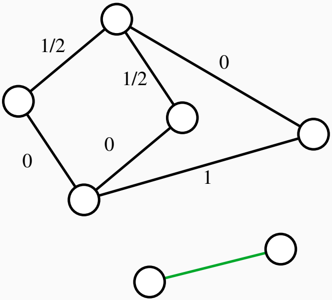

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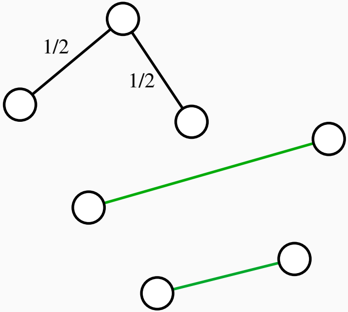

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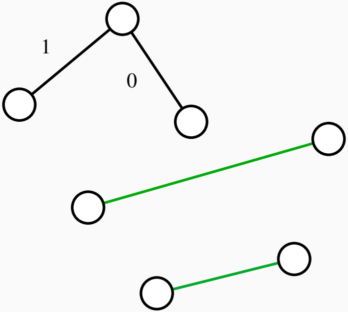

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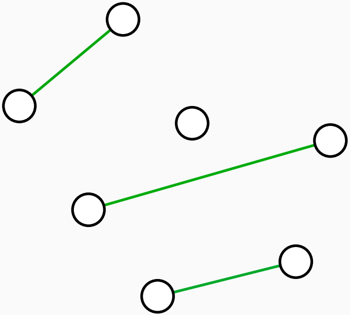
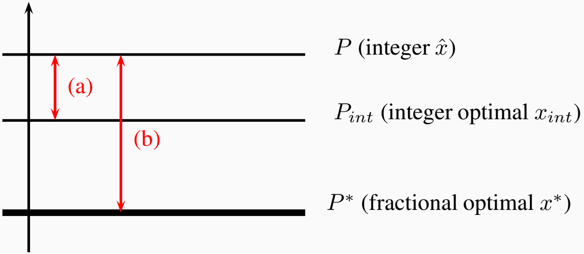


Integrality gap

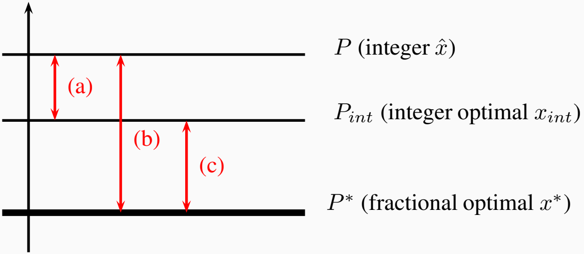
Integrality gap



(a) = Approximation ratio between $\hat{x}$ and x_{int}.

(b) = Approximation ration between $\hat{x}$ and x^*.

Integrality gap



(a) = Approximation ratio between $\hat{x}$ and x_{int}.

(b) = Approximation ration between $\hat{x}$ and x^*.

(c) = Integrality gap.